Problem set, Chapter 2: Arithmetic Functions and Dirichlet Multiplication

Damien Lefebvre

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1 6 Completed Exercise

Exercise 1.

a)

Let $n = 2^k$ with k integer be the prime factorization of every power of 2, since 2 is prime. So every power of 2 is relatively prime with odd numbers, or $(2^k, 2h+1) = 1$ for h integer.

For all n integer, there are $\frac{n}{2}$ odd numbers smaller than n.

Therefore, if $n = 2^k$, then all the relatively prime numbers to n are all the odd numbers smaller than n, or $\phi(n) = \frac{n}{2}$.

b)

Recall $\phi(n)$ is multiplicative. If $(2, n) = 1$, then

$$\phi(2n) = \phi(2)\phi(n) = 1\phi(n) = \phi(n) \tag{1}$$

So $\phi(n) = \phi(2n)$ for all $(2, n) = 1$, or for all n odd.

c)

$\phi(n) = 12$ for $n \in D$, with $D = \{13, 21, 26, 28, 36, 42\}$.

Exercise 2.

a)

Let $m = 3$ and $n = 4$ with $\phi(m) = \phi(3) = 2$ and $\phi(n) = \phi(4) = 2$. Then $(m, n) = (3, 4) = 1$, yet $(\phi(m), \phi(n)) = (2, 2) = 2$, so this is a counter example.

b)

Let $n = 15$, n is composite and $\phi(n) = \phi(15) = 8$. Yet $(n, \phi(n)) = (15, 8) = 1$, so this is a counter example.

c)

We assume $m > n$. Let D_p denote the list of primes divisors of m and n. If the same primes divide m and n, then $m = n * k$ for $k \in D_p$. Notice that $(k, m) = (k, n) = k$.

Then $n\phi(m) = m\phi(n) \rightarrow n\phi(n * k) = (n * k)\phi(n) \rightarrow \phi(n * k) = k\phi(n)$ but $\phi(n * k) = \phi(n)\phi(k)\frac{k}{\phi(k)} = k\phi(n)$. This completes the proof.

Exercise 5.

It is evident that $v(n)$ is a multiplicative function, so f is multiplicative too because it is the Dirichlet product of two multiplicative functions. Therefore, we need only to focus on $f(p^a)$ for p prime and a an integer.

$$f(p^a) = \sum_{d|p^a} \mu(d)v(\frac{p^a}{d}) \tag{2}$$

$$f(p^a) = \mu(1)v(\frac{p^a}{1}) + \mu(p)v(\frac{p^a}{p}) + ... + \mu(p^a)v(\frac{p^a}{p^a}) \tag{3}$$

$$f(p^a) = \mu(1)v(p^a) + \mu(p)v(p^{a-1}) + ... + \mu(p^a)v(1) \tag{4}$$

Notice that $v(p^a) = 1$ for all p prime and a integer.

$$f(p^a) = \mu(1) + \mu(p) + ... + \mu(p^a) * 0 \tag{5}$$

We know that $\mu(1) = 1$, $\mu(p) = -1$, and $\mu(p^a) = 0$ for all $a > 1$. Hence $f(p^a) = 1 - 1 = 0$ for all $a > 0$. If $a = 0$, then

$$f(p^a) = f(p^0) = f(1) = \sum_{d|1} \mu(d)v(\frac{1}{d}) = \mu(1)v(1) = 1 \tag{6}$$

If $a = 1$, then

$$f(p^a) = f(p) = \sum_{d|p} \mu(d)v(\frac{p}{d}) = \mu(1)v(p) + \mu(p)v(1) = 1 + 0 = 1 \tag{7}$$

So f(n) is either 0 or 1.

Exercise 7.

1. If $n = 1$, then

$$\sum_{d|n} \mu(d)\mu(p,d) = \sum_{d|1} \mu(d)\mu(p,d) = \mu(1)\mu(p,1) = \mu(1)^2 = 1^2 = 1 \tag{8}$$

2. If $n = p^a, a \geq 1$, then

$$\sum_{d|p^a} \mu(d)\mu(p,d) = \mu(1)\mu(p,1) + \mu(p)\mu(p,p) + ... + \mu(p^a)\mu(p,p^a) \tag{9}$$

If $a = 1$, then $\mu(p^a) = \mu(p) = -1$. If $a > 1$, then $\mu(p^a) = 0$. Also, if $a \geq 1$, then $\mu(p,p^a) = \mu(p) = -1$. Then

$$\sum_{d|p^a} \mu(d)\mu(p,d) = 1 + (-1) * (-1) = 1 + 1 = 2 \tag{10}$$

3. If $n = k^a, a \geq 1$ for k a prime distinct from p, then

$$\sum_{d|k^a} \mu(d)\mu(p,d) = \mu(1)\mu(p,1) + \mu(k)\mu(p,k) + ... + \mu(k^a)\mu(p,k^a) \tag{11}$$

Again, if $a = 1$ then $\mu(k^a) = \mu(k) = -1$. If $a > 1$, then $\mu(k^a) = 0$. Now, if $a \geq 1$ then $\mu(p, k^a) = 1$. Then

$$\sum_{d|k^a} \mu(d)\mu(p,d) = 1 + (-1) * 1 = 1 - 1 = 0 \tag{12}$$

4. If $n = \prod_{i=0}^{r} p_i^{a_i}$ for $r > 0$, then we have the same result as before.

Exercise 10.

Suppose there are k positive divisors of n, then $d(n) = k$. The following is the product of all k positive divisors of n.

$$\prod_{t|n} t \tag{13}$$

Assume n is square-free. Then all t go in pair, ie. $n = t_i * t_j$. If we multiply all k positive divisors of n, then we can rearrange them in pairs.

$$\prod_{t|n} t = (t_i * t_j) * (t_{i+1} * t_{j+1}) * ... * (t_{k-1} * t_k) \tag{14}$$

$$\prod_{t|n} t = n * n * ... * n \tag{15}$$

Since every t goes in pair, we can express the product as a power of n.

$$\prod_{t|n} t = n^{k/2} = n^{d(n)/2} \tag{16}$$

Exercise 11.

The positive divisors of a number n always go in pair, for instance $12 = 3 * 4$. Knowing that d(n) denotes the number of positive divisors of n, we can assume that d(n) is always even. However, if n is a square, then there is an additional, single positive divisor, for instance $36 = 6 * 6$. Then if, and only if, n is a square, then d(n) is odd.

2 2 Incomplete Exercises

Exercise 3.

We claim

$$\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\phi(d)} = \sum_{d|n} \frac{\mu(d)}{\phi(d)} * \mu(d) \tag{17}$$

We use Mobius inversion

$$\frac{\mu(n)}{\phi(n)} = \sum_{d|n} \frac{d}{\phi(d)} \tag{18}$$

Exercise 4.

The first 8 primes are $D = \{2, 3, 5, 7, 11, 13, 17, 19\}$. Then the first number n with 8 distinct prime factors (DPF) is the product of all elements in D:

$$n = \prod D = 2 * 3 * 5 * 7 * 11 * 13 * 17 * 19 = 9699690 \tag{19}$$

We can also see that

$$\phi(n) > n/6 \tag{20}$$

$$6\phi(n) > n \tag{21}$$

$$6\phi(n) > \sum_{d|n} \phi(d) \tag{22}$$